Bisymmetric Tutte Polynomial Generalizations

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Both the Stanley Symmetric Chromatic Function X_G and the Tutte Polynomial T_G are well-known and successful objects for graph theory. Here, we consider an object that incorporates and generalizes properties of both. In particular, we focus on the duality property of T_G, namely that $T_{G^*}(x, y) = T_G(y, x)$.

We work with functions f that are symmetric in two sets of countably many variables, $(x_1, x_2, ...)$ and $(y_1, y_2, ...)$, where f is invariant under permutations of the x variables and the y variables separately; we call such a function *bisymmetric*. However, we will almost exclusively work with the power sum symmetric function bases $p_k = \sum x_i^k$ and $q_k = \sum y_i^k$, and functions written in terms of these bases are automatically bisymmetric. We write $p = (p_1, p_2, ...)$ and $q = (q_1, q_2, ...)$, so that a standard bisymmetric function is of the form $f(p, q)$.

Motivation for a new object: the *Sesquigraph*

To better consider the notion of a graph and its dual at once, we introduce *sesquigraphs*. Our initial need for a new concept sprang from two observations: one in the deletion-contraction process, and one in graph-link equivalence. The deletion-contraction process is discussed in the next section. There is a well-known method to map alternating links to graphs that equates vertices with link regions and edges to crossings [1]. By choosing whether or not the external region is colored, two different colorings, and subsequently two different graphs, are associated to an alternating link. In many cases, these two graphs are planar duals, but this is not always the case.

For example, compare the two possible graphs associated with the unknot to the two graphs associated with two concentric unknots. Fixing the color of the external region to be associated with the "dual" graph, we convert these two link diagrams to graphs in the usual way. The graph associated with an unknot is a vertex whose dual is a single co-vertex. On the other hand, the graph associated with two concentric unknots is a vertex whose dual is two co-vertices. Hence, without knowledge of the link, a graph does not have enough information to determine its dual. Sesquigraphs resolve this problem by pairing the graph with its dual.

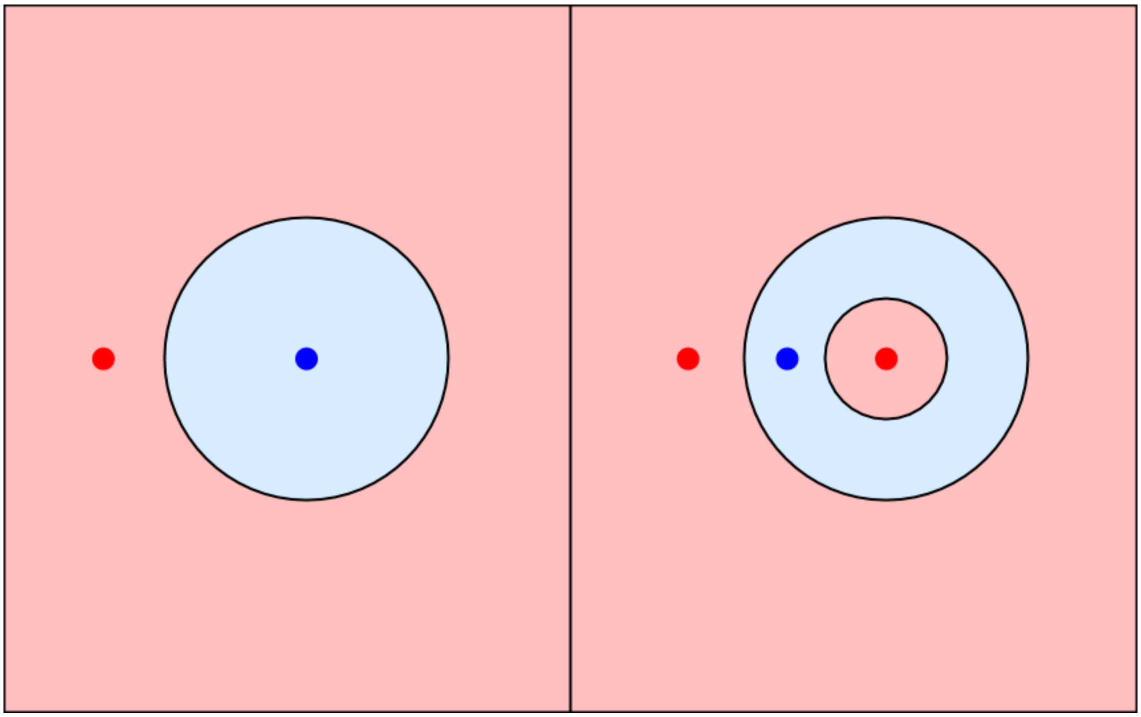

Figure 1: Here, blue vertices are part of principal graphs, while red vertices are part of dual graphs. Under the standard coloring, where the outside region is given a co-vertex, note that the graph of a single vertex has varying duals (here, a single co-vertex vs. two co-vertices), depending on the original link diagram.

Sesquigraphs and Deletion-Contraction

A sesquigraph is a pair of 2 graphs whose edge count is the same; a sesquigraph $G = \langle A, B, E \rangle$ can be produced from two graphs (A, E_1) and (B, E_2) by placing E_1 and E_2 in a bijection and constructing E to be the set of all pairs of edges in bijection. This definition necessarily includes a choice of bijection, so many different sesquigraphs can be created from a pair of graphs.

Usually sesquigraphs are constructed from a graph and its dual, so we call $G_A = (A, E)$ the *principal* graph and $G_B = (B, E)$ the *dual* or *secondary* graph, and elements of E are still called edges. When elements of E are invoked on G_A or G_B, only the relevant edge of the bijection is considered. Deletion and contraction on sesquigraphs can be performed on any edge (even if it was a loop or a bridge in one of the original graphs) as follows.

Definition 1. The deletion of G by an element $e \in E$ is

$$G - e = \langle A, B/e, E - e \rangle$$

Definition 2. The contraction of G by an element $e \in E$ is

$$G/e = \langle A/e, B, E - e \rangle$$

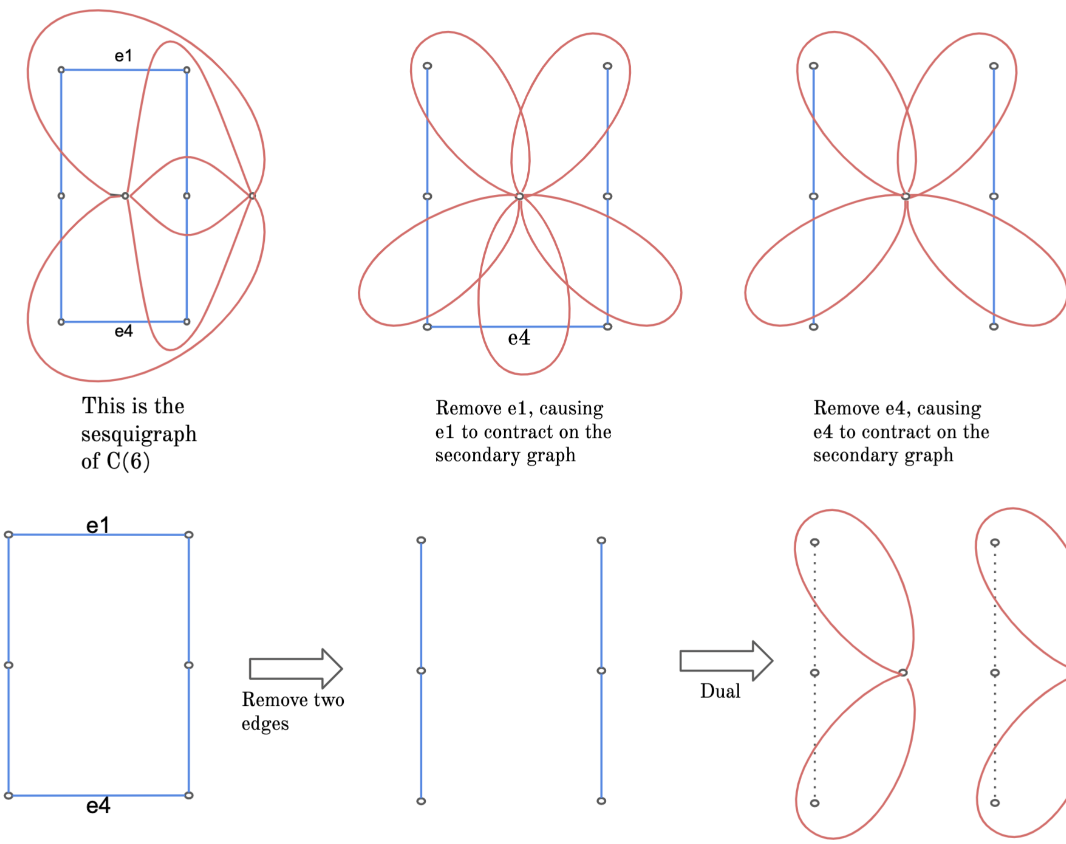

Figure 2: A cycle on 6 vertices (blue) and its dual (red) are used to create a sesquigraph in two different ways. In the first row, the standard sesquigraph is constructed and then two deletion-contractions are performed. In the second row, deletion of edges is done before the construction of the dual. Information about the original graph's connected components is lost.

For a standard sesquigraph (constructed from a graph, its dual, and the choice of the obvious bijection), deletion and contraction differ slightly from the standard notions. Most importantly, contracting a loop and deleting a loop are now two different operations with different results. Namely, contracting a loop on A does not change either vertex set, but deleting a loop on A contracts the corresponding bridge in B and hence the vertex set B is changed. Thus contracting loops on A deletes bridges on B, which changes the number of connected components on B.

Definition 3. The *Dual-symmetric Tutte Polynomial* D_G for a sesquigraph constructed from two weighted graphs is defined by the following deletion contraction procedure:

1. $D_G(p,q) = D_{G/e}(p,q) + D_{G-e}(p,q)$ for any $e \in E$
2. $D_G(p,q) = \prod_{a \in A} p_{w(a)} \prod_{b \in B} q_{w(b)}$ if G has no edges

When vertices are contracted, their weights are summed, as in a standard weighted graph. The function $w(a)$ returns the weight of a vertex a.

Properties of D_G

From the deletion contraction defintion of D_G, we state the general formula.

Theorem 1.

$$D_G(p,q) = \sum_{F \subseteq E(G)} \prod_{S \in \kappa(A,F)} p_{W_A(S)} \prod_{S' \in \kappa(B,\overline{F})} q_{W_B(S')} \tag{1}$$

Proof. The proof essentially follows from induction on the number of vertices and deletion contraction.

If there are no edges in E(G), then the only subset of it is $\emptyset$, so the summation reduces to part 2 of definition 3, and the base case is established.

Now assume for the sake of induction that graphs with cardinality k do follow this formula. Then, consider a graph with k + 1 edges:

$$D_G(p,q) = D_{G-e}(p,q) + D_{G/e}(p,q)$$

By the hypothesis,

$$D_G(p,q) = \sum_{F\subseteq E(G-e)} \prod_{S\in\kappa(A,F)} p_{W_A(S)} \prod_{S'\in\kappa(B,\overline{F})} q_{W_B(S')} + \sum_{F\subseteq E(G/e)} \prod_{S\in\kappa(A,F)} p_{W_A(S)} \prod_{S'\in\kappa(B,\overline{F})} q_{W_B(S')}$$

We can map $F \subseteq E(G-e)$ to a subset that does not contain e, while we can map $F \subseteq E(G/e)$ to a subset that does contain e.

It is verified easily that considering them in this way does not change the products on the inside of the sum, and since every subset either contains e or doesn't contain e, we get that

$$D_G(p,q) = \sum_{F\subseteq E(G)} \prod_{S\in\kappa(A,F)} p_{W_A(S)} \prod_{S'\in\kappa(B,\overline{F})} q_{W_B(S')}$$

As desired. □

Now we show that D_G is dual symmetric.

Theorem 2. $D_G(p,q) = D_{G^*}(q,p)$

Proof. We once again proceed using induction.

In a sesquigraph, taking the dual simply corresponds to switching the principal and secondary sets, so for an empty graph, we get

$$D_{G^*}(p,q) = \prod_{b\in B} p_{w(b)} \prod_{a\in A} q_{w(a)}$$

Which is equal to $D_G(q,p)$

Now assume through induction that it is true for graphs with k vertices. Then take a graph with k + 1 vertices:

$$\begin{aligned} D_G(p,q) &= D_{G-e}(p,q) + D_{G/e}(p,q) \\ &= D_{(G-e)^*}(q,p) + D_{(G/e)^*}(q,p) \\ &= D_{G^*/e}(q,p) + D_{G^*-e}(q,p) \\ &= D_{G^*}(q,p) \end{aligned}$$

Thus, we are done. □

Theorem 3. *D_G specializes to the dichromatic polynomial Z_G and Stanley's symmetric function X_G as follows:*

1. *With the specialization of $p_i = t/u$ and $q_i = u$ for all $i \leq n$, we have that*

$$D_G(p,q) = u^{k(G)-v(G)} Z_G(t,u)$$

2. *Here, let -1 be the sequence $q_i = -1$. Then,*

$$D_G(p,-1) = (-1)^{1-V(G)} X_G(p)$$

Proof. Both results follow from the definitions of Z_G and X_G. □

A Convolutional Definition

Let $G = \langle A, B, E\rangle$ be a weighted sesquigraph and let $p = [p_1, p_2, \ldots]$ and $q = [q_1, q_2, \ldots]$.

Remark. The Tutte polynomial can be written convolutionally as [2]

$$\begin{aligned}
T_G(1-x, 1-y) &= \sum_{S\subseteq E} T_{G/S}(1-x, 0) T_{G|S}(0, 1-y) \\
&= \sum_{S\subseteq E} T_{G/S}(1-x, 0) T_{G^*/(E-S)}(1-y, 0) \\
&= \sum_{S\subseteq E} x^{-k(G/S)}(-1)^{|G/S|+k(G/S)} \chi_{G/S}(x)(-1)^{|H_S|+k(H_S)} y^{-k(H_S)} \chi_{H_S}(y) \\
&= x^{-k(G)} y^{-k(H_S)} \sum_{S\subseteq E} (-1)^{|G/S|+k(G/S)+|H_S|+k(H_S)} \chi_{G/S}(x)\chi_{H_S}(y) \\
&= (-x)^{-k(G)}(-y)^{-k(G^*)} \sum_{S\subseteq E} (-1)^{|G/S|+|H_S|} \chi_{G/S}(x)\chi_{H_S}(y)
\end{aligned}$$

where $H_S = G^*/(E-S)$ and $\chi_G(x)$ is the chromatic polynomial evaluated on a graph G.

This motivated the definition of a polynomial that could generalize the Tutte polynomial and the Stanley Symmetric Chromatic polynomial.

Definition 4. The *Symmetric Tutte Polynomial* is defined by

$$C_G(p,q) = \sum_{S\subseteq E} (-1)^{|A/S|+|B/(E-S)|} X_{A/S}(p) X_{B/(E-S)}(q)$$

where $X_G = X_A$ is the Weighted Stanley Symmetric Chromatic Function of G [3] evaluated on the principal graph A), and $X_{G^*} = X_B$. As it turns out, this definition gives rise to the same polynomial as the deletion-contraction definition.

Lemma 4. $C_G = C_{G-e} + C_{G/e}$

Proof. First observe that if $e \in S \subseteq E$, then the following identities hold:

1. $A/S = (A/e)/(S-e)$
2. $B/(E-S) = B/((E-e)-(S-e))$

and, if $e \notin S \subseteq E$, then the following identity holds as well:

3. $B/(E-S) = (B/e)/((E-e)-S)$

Now we see that

$$\begin{aligned}
C_G(p,q) &= \sum_{S\subseteq E} (-1)^{|A/S|+|B/(E-S)|} X_{A/S}(p) X_{B/(E-S)}(q) \\
&= \sum_{e\in S\subseteq E} (-1)^{|A/S|+|B/(E-S)|} X_{A/S}(p) X_{B/(E-S)}(q) \\
&\quad + \sum_{e\notin S\subseteq E} (-1)^{|A/S|+|B/(E-S)|} X_{A/S}(p) X_{B/(E-S)}(q) \\
&= \sum_{S-e\subseteq E-e} (-1)^{|(A/e)/(S-e)|+|B/([E-e]-(S-e))|} X_{(A/e)/(S-e)}(p) X_{B/([E-e]-(S-e))}(q) \\
&\quad + \sum_{S\subseteq E-e} (-1)^{|A/S|+|(B/e)/([E-e]-S)|} X_{A/S}(p) X_{(B/e)/([E-e]-S)}(q) \\
&= C_{G/e}(p,q) + C_{G-e}(p,q)
\end{aligned}$$

□

Theorem 5. $C_G(p,q) = D_G(-p,-q)$

Proof. One can easily check that for point clouds (i.e. $E = \emptyset$) both functions are the same. Moreover, $D_G(-p,-q)$ satisfies the deletion-contraction recurrence, so as long as $C_G(p,q)$ satisfies the same deletion-contraction—which it does by the previous lemma—the functions must be equal. □

A Direct-Symmetric Function Definition

For shorthand, we write $x^{\kappa(A)} = \prod_{a \in A} x_{\kappa(a)}$ and $y^{\kappa(B)} = \prod_{b \in b} y_{\kappa(b)}$.

Theorem 6. *Let G be an unweighted sesquigraph. Let $\kappa : A \times B \to \mathbb{N}$ be a coloring (not necessarily proper) of both A and B, and let C_κ be the number of induced subsesquigraphs of κ that keep every component of both A and B monochromatic. Then*

$$D_G(x, y) = \sum_{\kappa} C_\kappa x^{\kappa(A)} y^{\kappa(B)}$$

where κ ranges over all colorings of both A and B.

Proof. The proof exactly mirrors that of [4, Theorem 2.5]. □

General Properties of Bisymmetric Specializations

We now turn to general properties of bisymmetric functions $f_G(p, q)$ that *specialize* to both T_G and X_G, with the *dual symmetric* property $f_{G^*}(p, q) = f_G(q, p)$. We will quickly find that, for reasonable definitions of specialize, any such object will be embedding dependent, and cannot easily specialize to a link invariant without dropping the indices.

A good definition of specialization is difficult to formulate [5], but the desired notion is that a specialization includes substituting expressions in for each p_i, say $s_{p_i}(p_i^k, G)$ and $s_{q_i}(q_i^k, G)$ (naturally these substitution functions should distribute across sums and products) and then doing some general transformations $h(f_G(s_p(p), s_q(q)), G)$ to the function as a whole.

We immediately notice this definition does not prevent h from ignoring f entirely and being dependent only on G; rather than linger on this suboptimal definition, we will instead only discuss the subject informally and paint a picture for how much information from G is needed for the specialization. Ideally, a specialization would use no more information than the number of vertices, edges, faces, and connected components of the graph. We will refer to such attributes of a graph as *global* information, and will call a specialization *reasonable* if it only relies on such information. We bring attention to the fact that global information alone is not enough to distinguish any two trees with the same number of edges.

Claim 1. Any function $F_G(p, q)$ that is dual symmetric and reasonably specializes to X_G must be embedding dependent.

(Informal) Proof. Consider two trees T_1 and T_2 both with m edges. Their duals T_1^* and T_2^* are both a single vertex with m loops, and hence their only difference is their embedding. If $F_G(p, q)$ is embedding independent, then

$$F_{T_1}(p, q) = F_{T_1^*}(q, p) = F_{T_2^*}(q, p) = F_{T_2}(p, q).$$

We observe that T_1 and T_2 completely share their global information, but that there are trees T_1 and T_2 that X_G can distinguish. Hence any specialization from F_G to X_G would have to rely on enough information to distinguish these trees, and hence is not reasonable. □

When initially studying this function, we were hoping to use it to construct a link invariant similar to how the Tutte Polynomial can be used to construct the Bracket Polynomial [1]. However, we run into issues with trees again.

Claim 2. Any symmetric function $F_G(p)$ that reasonably specializes to X_G cannot specialize to a link-invariant without confounding all variables p_i and q_i.

Informal Argument. First notice that every tree represents the same link - the unknot. Hence every tree must map to the same value for the object to be a link invariant. However, you can construct many polynomial relationships between the variables p_i as follows:

Consider a path graph $A = P(n)$ and an "almost-path" graph B that is $P(n-1)$ with a leaf attached, by an edge e, $k \leq n$ vertices away from one of the ends of the path. Then applying deletion-contraction to edge e on B and to the edge k vertices from the bottom on A, the contracted graph is the same for both graphs. However, this is not true for edge deletion. $B-e$ has an isolated vertex and a path $P(n-1)$. $A-e$ is the two paths $P(n-k)$ and $P(k)$. Since both original graphs are trees, both must specialize to the same link invariant, and hence the two graphs resulting from the deletions must specialize to the same invariant as well. However, this gives strict polynomial relationships between what each variable p_i can specialize to. We would be surprised if these equations did not force all the variables p_i to specialize to the same thing. □

References

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